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A Computational Method for Determining Strong Stabilizability of n-D Systems

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This paper describes an algorithm for the following problem: given two multivariate complex or real polynomials f and g, decide whether there exist complex or real polynomials h and k such that both k and $fh + gk$ have no zero in the unit polydisc. This problem, known as strong stabilizability, is fundamental in control theory, with important applications in designing stable feedback systems with a stable compensator. Our algorithm for solving the problem is formulated based on the *cylindrical algebraic decomposition* (cad) of an algebraic variety. While recent applications of cad to systems and control have been focused on those problems which have a quantifier elimination formulation, our method is novel in that it explicitly computes some topological properties of an algebraic variety based on the cad to solve the problem for which a quantifier elimination formulation is not readily available.

1. Introduction

Many problems in control system analysis and design can be formulated as decision problems: is a given system stable? Is a given unstable system stabilizable? Under what condition does a system satisfy certain performance constraints? Some of these problems can be formulated as decision problems in the first order theory of real closed fields, and can be solved via quantifier elimination methods. See, e.g., Anderson *et al.* (1975), Bose (1982), Bose and Modarressi (1976), Dorato *et al.* (1997), Hong *et al.* (1997), Jirstrand (1997), Blondel and Tsitsiklis (1997). While the early research by Anderson, Bose *et al.* had, above all, a theoretic flavor, recent applications of algebraic decision methods to control system problems have begun to see real implementation, due to the growing power of computers and the development of computational methods and fast software (Collins, 1975; Collins and Hong, 1991). There is no doubt that algebraic decision methods will become increasingly important as tools to solve a wide variety of control system decision problems which have a quantifier elimination formulation. Unfortunately, not all the decision problems have such a formulation.

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For example, the question on the existence of a common compensator that simultaneously stabilizes three (or more) given conventional linear plants, a long standing problem in control system theory, seems not to have a simple quantifier elimination formulation (Blondel and Gevers, 1994). No effective computational procedure that solves this question has been found.

This problem is an extension of the two-plant simultaneous stabilization problem. It is well known that two linear systems can be simultaneously stabilized by a compensator if a certain other system can be stabilized by a *stable* compensator (see, e.g., Vidyasagar (1985, Chap. 5)). The later condition, referred to as *strong stabilizability*, is equivalent to the so-called *parity interlacing property* (Youla *et al.*, 1974) for a conventional linear system, which can be formulated as a simple quantifier elimination problem and be solved by rational operations. See the end of Section 2 of this paper for a discussion. In this paper, we will consider a problem in another direction of extension—strong stabilization of *multidimensional* systems. In purely mathematical terms, the problem can be described as follows.

PROBLEM. Given two multivariate complex or real polynomials $f(z)$ and $g(z)$, $z = (z_1, \ldots, z_n)$, decide whether there exist complex or real polynomials $h(z)$ and $k(z)$ such that

$$k(z) \neq 0 \qquad \text{and} \qquad f(z)h(z) + g(z)k(z) \neq 0, \qquad \forall z \in \bar{U}^n, \tag{1.1}$$

where $\bar{U}^n = \{z \in \mathbb{C}^n \,||z_1| \leq 1, \ldots, |z_n| \leq 1\}$ denotes the unit polydisc in $\mathbb{C}^n$.

Although we are not able to find a quantifier elimination formulation, we develop a novel computational procedure based on the topological structure of the *cylindrical algebraic decomposition* (Collins, 1975; Arnon *et al.*, 1984) for solving the problem.

The following paragraphs explain briefly a control system theoretic background for the problem addressed above, which may be skipped without loss of logical continuity of this paper. For an extensive background of multidimensional systems, the reader is referred to (Bose, 1977, 1982, 1985).

A conventional system is said to be one-dimensional (1-D) in the sense that it describes the relationship between input and output functions in a single time dimension. It is well known that a 1-D shift invariant (meaning that the property of the system is invariant with a time shift) linear system can be described by a rational transfer function, which is obtained by, e.g., Laplace transform or z transform. In contrast, an n-dimensional (n-D) system describes the relationship between input and output functions in n free variables, which may include temporal and spatial variables. Such a system is said to be shift invariant if its property is invariant with a shift in any of its free variables. An n-D linear shift invariant single-input single-output (SISO) system can be described by a rational transfer function $p(z) = f(z)/g(z)$, where f and g are relatively prime polynomials in the variables $z = (z_1, \ldots, z_n)$ with real or complex coefficients. In the following we call p a *complex system* when f and g are complex polynomials, in contrast we call it a *real system* if f and g are real polynomials.

The system $p(z)$ is by definition **stable** iff $p(z)$ is holomorphic in the unit polydisc $\bar{U}^n$, or equivalently, $g(z)$ is free from 0 in $\bar{U}^n$ (Jury, 1978; Bose, 1982).

In the case that the system p is not stable, using a standard feedback configuration shown in Figure 1 (Bose, 1985; Vidyasagar, 1985; Shankar and Sule, 1992), we have a feedback system with inputs u_1, u_2, and outputs y_1, y_2. In the figure, c denotes a

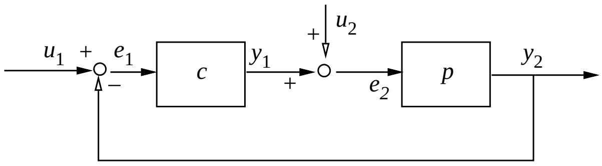

Figure 1. Feedback system.

compensator with transfer function $c(z) = \frac{h(z)}{k(z)}$, where $h(z)$ and $k(z)$ are relatively prime polynomials. The overall input–output relation can be written as:

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} \frac{c}{1+pc} & \frac{-pc}{1+pc} \\ \frac{pc}{1+pc} & \frac{p}{1+pc} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} \frac{hg}{kg+hf} & \frac{-hf}{kg+hf} \\ \frac{hf}{kg+hf} & \frac{kf}{kg+hf} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}. \tag{1.2}$$

The closed loop feedback system is stable iff each entry of the transfer matrix is holomorphic in $\bar{U}^n$. It is easy to see that this condition is equivalent to that

$$fh + gk \neq 0 \text{ in } \bar{U}^n. \tag{1.3}$$

It is also known (Shankar and Sule, 1992) that the condition that f and g have no common zero in $\bar{U}^n$ is a necessary and sufficient condition for the existence of a compensator $c(z) = \frac{h(z)}{k(z)}$ such that (1.3) holds. In this case p is said to be **stabilizable**. If a stabilizable system p is real, that is, it possesses a transfer function with real coefficients, then the compensator c can also be chosen to be real (Shankar and Sule, 1992, Theorem 3.1.21).

If the compensator c itself can be chosen to be stable, then p is said to be **strongly stabilizable**. To determine the strong stabilizability of a plant $p = f/g$, we have to solve the mathematical problem (1.1) described above. In Shankar (1994), a topological criterion is derived to determine the existence of a stable complex compensator c for a given system p. When p is a real system, a necessary and sufficient condition for the existence of a stable *real* compensator for p is given by Ying (1998). Both of these conditions involve the topological and geometric structures of the algebraic variety $V(f) \cap \bar{U}^n$. In Ying (1998), the computational problems for determining these conditions are defined and solutions based on the cylindrical algebraic decomposition are suggested.

The contribution of this work is the development of an efficient computational method for testing the strong stabilizability conditions. The rest of this paper is organized as follows. In the next section these conditions are stated without proofs, which can be found in Ying (1998). In Section 3, a novel computational procedure for testing these conditions is presented. In Section 4 an example is worked out to illustrate the procedure for computing a winding number. A brief summary of this work along with some open problems are described in Section 5. Two appendices are included at the end to clarify some facts claimed in the paper.

2. Conditions for Strong Stabilizability

In this section we describe two main mathematical theorems that give computable conditions for the strong stabilizability problem. Let $f(z)$ and $g(z)$ be two complex or real polynomials. Let

$$V(f) = \{(z_1, \ldots, z_n) \in \mathbb{C}^n |\ f(z_1, \ldots, z_n) = 0\},$$

$$V(g) = \{(z_1, \dots, z_n) \in \mathbb{C}^n | \; g(z_1, \dots, z_n) = 0\}$$

be two complex varieties defined by f and g, respectively. Throughout this paper we assume that $V(f) \cap V(g) \cap \bar{U}^n = \emptyset$. This is equivalent to the property that the system described by the transfer function f/g is stabilizable, but not necessarily strongly stabilizable.

In Section 2.1, the condition for the existence of two complex polynomials $h(z)$ and $k(z)$ such that $k(z) \neq 0$ and $f(z)h(z) + g(z)k(z) \neq 0, \forall z \in \bar{U}^n$ is given. The further condition for that $h(z)$ and $k(z)$ can be chosen to be real when $f(z)$ and $g(z)$ are real is described in Section 2.2. Their implications in feedback system design are also briefly explained following the mathematical theorems.

2.1. STABILIZATION BY STABLE COMPLEX COMPENSATORS

Suppose that $h(z)$ and $k(z)$ are already given such that

$$k(z) \neq 0 \qquad \text{and} \qquad g(z)k(z) + f(z)h(z) \neq 0, \qquad \forall z \in \bar{U}^n.$$

Then we have

$$u(z) = g(z) + f(z)\frac{h(z)}{k(z)} \neq 0, \qquad \forall z \in \bar{U}^n.$$

As $\bar{U}^n$ is simply connected, $u(z)$ has a single-valued logarithmic function defined by

$$\log u(z) = \log u(0) + \int_\gamma \frac{du(z)}{u(z)} = \log u(0) + \int_{u(\gamma)} \frac{d\tau}{\tau},$$

where γ is a path connecting the origin to a point z in $\bar{U}^n$, τ is a complex variable.

As a consequence, $\log g(z)$ is defined to be *single-valued* over $V(f) \cap \bar{U}^n$. This implies that for any closed cycle $\gamma \subset V(f) \cap \bar{U}^n$, we have

$$W(g(\gamma)) = \frac{1}{2\pi i} \int_{g(\gamma)} \frac{d\tau}{\tau} = 0. \tag{2.1}$$

Recall that this is the winding number of the closed curve $g(\gamma)$ around the origin in $\mathbb{C}^* = \mathbb{C}\backslash\{0\}$. See, e.g. Fulton (1995). The converse of this fact also holds.

THEOREM 2.1. (YING, SHANKAR) *Let $f(z)$ and $g(z)$ be two complex polynomials. Then a necessary and sufficient condition for the existence of two complex polynomials $h(z)$ and $k(z)$ such that*

$$k(z) \neq 0 \qquad \textit{and} \qquad f(z)h(z) + g(z)k(z) \neq 0, \qquad \forall z \in \bar{U}^n$$

is that every closed curve γ on $V(f) \cap \bar{U}^n$ is mapped by g to a closed curve $g(\gamma)$ in $\mathbb{C}^* = \mathbb{C}\backslash\{0\}$ whose winding number around the origin is equal to zero

$$W(g(\gamma)) = \frac{1}{2\pi i} \int_{g(\gamma)} \frac{d\tau}{\tau} = 0.$$

COROLLARY 2.1. *A system $p(z) = \frac{f(z)}{g(z)}$ can be stabilized by a stable complex compensator iff every closed curve γ on $V(f) \cap \bar{U}^n$ is mapped by g to a closed curve $g(\gamma)$ in $\mathbb{C}^* = \mathbb{C}\backslash\{0\}$ whose winding number around the origin is equal to zero.*

If γ and γ' are two homologous cycles on $V(f) \cap \bar{U}^n$, then $g(\gamma)$ and $g(\gamma')$ are homologous, and so have the same winding number around the origin in $\mathbb{C}^*$. The condition of Theorem 2.1 is equivalent to the condition that a representative of each of the first-order homology classes of $V(f) \cap \bar{U}^n$ is mapped by g to a curve in $\mathbb{C}^*$ with winding number equal to 0.

EXAMPLE 1. Let $f = 1 - 4z_1z_2$, $g = z_1$. The curve γ defined by

$$\gamma : t \longrightarrow \left(\frac{1}{2}e^{i2\pi t}, \frac{1}{2}e^{-i2\pi t}\right), \qquad t \in [0,1]$$

is mapped by g to a cycle $\{\frac{1}{2}e^{i2\pi t},\ 0 \le t \le 1\}$ in $\mathbb{C}^*$, whose winding number around the origin is equal to 1. Thus the system f/g is not strongly stablizable. See Sections 3 and 4 for details on how to compute the winding number of a cycle.

EXAMPLE 2. Let $f = 1 - 4z_1z_2$, $g = 1 + z_1 - z_2$. The curve γ defined by

$$\gamma : t \longrightarrow \left(\frac{1}{2}e^{i2\pi t}, \frac{1}{2}e^{-i2\pi t}\right), \qquad t \in [0,1]$$

is mapped by g to a curve $\{1 + i\sin 2\pi t,\ 0 \le t \le 1\}$ in $\mathbb{C}^*$, whose winding number around the origin is equal to 0. It is easy to see that γ generates the first homology group of $V(f) \cap \bar{U}^n$. Thus the system is strongly stablizable. Indeed, using the *Gröbner bases* method (Buchberger, 1985; Xu *et al.*, 1994), we can find a stable compensator $c(z) = h(z)/k(z) = \frac{1}{2(1+\sqrt{2})}$ which stabilizes f/g. See Appendix A.

2.2. STABILIZATION BY STABLE REAL COMPENSATORS

Let $p(z) = \frac{f(z)}{g(z)}$, $f, g \in \mathbb{R}[z_1, \ldots, z_n]$. Let X be a connected component of $V(f) \cap \bar{U}^n$. X is said to be *self-conjugate* if X is identical to its conjugate $\bar{X}=\{z | \overline{z} \in X\}$.

DEFINITION 2.1. Let X be a self-conjugate connected component of $V(f) \cap \bar{U}^n$, g is said to have a *positive sign* on X if there exists a holomorphic function G on X such that

$$e^{G(z)} = g(z) \qquad \text{and} \qquad \overline{G(z)} = G(\overline{z}) \qquad \forall z \in X; \tag{2.2}$$

g is said to have a *negative sign* on X if $-g$ have a positive sign on X.

REMARK. If z_0 is a real point on X, then X is self-conjugate and the above definition coincides with the sign of the value $g(z_0)$. See Appendix B.

PROPOSITION 2.1. (YING) *Let γ_0 be a path in X connecting a point $z_0 \in X$ to its conjugate $\overline{z_0}$. Let G_0 be a complex (possibly real) number such that $e^{G_0} = g(z_0)$. Then*

$$\int_{\gamma_0} \frac{dg}{g} = \overline{G_0} - G_0 + m2\pi i, \tag{2.3}$$

where m is an integer. Moreover, g has a positive sign if m is an even number, and g has a negative sign if m is an odd number.

Theorem 2.2. (Ying) *Let $f(z)$ and $g(z)$ be two real polynomials which already satisfy the condition of Theorem 2.1. Then a necessary and sufficient condition for the existence of two real polynomials $h(z)$ and $k(z)$ such that*

$$k(z) \neq 0 \qquad \text{and} \qquad f(z)h(z) + g(z)k(z) \neq 0, \qquad \forall z \in \bar{U}^n$$

is that $g(z)$ has a constant sign (either $+$ or $-$) over the union of self-conjugate connected components of $V(f) \cap \bar{U}^n$.

Corollary 2.2. *A strongly stabilizable real system $p(z) = \frac{f(z)}{g(z)}$ is stabilizable by a stable real compensator iff $g(z)$ has a constant sign (either $+$ or $-$) over the union self-conjugate connected components of $V(f) \cap \bar{U}^n$.*

The following example is to illustrate the case where both a complex stable compensator and a real unstable compensator exist, but a real stable compensator does not exist.

Example 3. $f = z_1^2 - z_2 - 2$, $g = z_1$. As

$$\bar{U}^2 \cap V(f) = \{(-1, -1), (1, -1)\}$$

is a discrete point set, the condition for strong stabilizability of Theorem 2.1 is trivially satisfied. By trial, we found a complex stable stabilizing compensator $c = \frac{-1}{0.5z_1+i}$. In fact, it is not difficult to check that the polynomial

$$(0.5z_1 + i)g - f = -0.5z_1^2 + iz_1 + z_2 + 2$$

is free from 0 in $\bar{U}^2$. However, g has opposite signs at the two discrete points of $\bar{U}^2 \cap V(f)$. Hence there is no real stable stabilizing compensator. Using the Gröbner bases method, we can find a real compensator $\frac{-1}{z_1}$, which is not stable.

In the 1-D case, this sign consistency condition is equivalent to the "parity interlacing property" by Youla *et al.* See Youla *et al.* (1974), Vidyasagar (1985) and Ying (1998). This condition can be tested by eliminating the quantifiers in the following formula:

$$\forall x(x \in I \wedge f(x) = 0 \rightarrow g(x) > 0) \vee \forall x(x \in I \wedge f(x) = 0 \rightarrow g(x) < 0),$$

where $I = [-1, 1]$ or $I = [0, +\infty)$ (Ying, 1998).

3. Computational Procedures for Determining Strong Stabilizability

In this section we present computational procedures for testing the conditions of Theorems 2.1 and 2.2 based on the *Cylindrical Algebraic Decomposition* of a semi-algebraic set ("cad" hereafter, see Collins (1975), Arnon *et al.* (1984)).

3.1. cad and homology

Given a finite set A of d-variate polynomials with real coefficients, an A-*invariant cad* of $\mathbb{R}^d$ consists of a finite number of disjoint cells, where a cell is by definition a connected set homeomorphic to a real r-dimensional open unit ball, such that each of the given polynomials has a constant sign ($= 0$, > 0 or < 0) on each cell. An r-dimensional cell is called an r-cell. For later use, here we briefly address the description of a 1-cell. Precise description of general cells can be found in the literature.

A 1-cell in the cylindrical algebraic decomposition is a space curve parametrized by some coordinate x_r, and is defined by the following formulae:

$$\begin{cases} x_1 = a_1, \\ \dots, \\ x_{r-1} = a_{r-1}; \\ a_r \le x_r \le b_r; \\ F_{r+1}(a_1, \dots, a_{r-1}, x_r, x_{r+1}) = 0, \\ \dots, \\ F_d(a_1, \dots, a_{r-1}, x_r, x_{r+1}, \dots, x_d) = 0. \end{cases} \tag{3.1}$$

In general there are multiple 1-cells that satisfy the above constraints. But they can be discriminated by a certain natural order (Collins, 1975; Arnon *et al.*, 1984).

A semi-algebraic set is a subset of $\mathbb{R}^d$ whose coordinates satisfy a finite number of polynomial equalities and inequalities. Let A denote the set of polynomials involved in the definition of a semi-algebraic set. The semi-algebraic set can be identified with a union of cells in an A-invariant cad. We call such a union a cad of the semi-algebraic set.

The topological characteristics, such as the homology groups, of a semi-algebraic set can be computed by decomposing it into a standard topological structure called *regular cell complex*. See Cooke and Finney (1967), Schwartz and Sharir (1983) for a precise definition. In a regular cell complex, the adjacency relations between cells can be described by an *incidence function* α on cell pairs. For two cells c and c', $\alpha(c, c') = 1$ or -1 if c' belongs to the boundary of c and is of dimension exactly one less than the dimension of c, $\alpha(c, c') = 0$ otherwise. This function should satisfy some other rules. See Cooke and Finney (1967), Schwartz and Sharir (1983) for details. It is essential for us that, with possibly a proper linear transformation of coordinates, a cad can be constructed to form a regular cell complex, and an incidence function can be assigned to the cad so that the homology groups of the semi-algebraic set can be computed from the cad (Schwartz and Sharir, 1983, Theorem 3).

We identify $V(f) \cap \bar{U}^n$ as a semi-algebraic subset of R^{2n} with a proper real coordinate system $\{x_1, \dots, x_{2n}\}$ so that the cad of $V(f) \cap \bar{U}^n$ is a regular cell complex.

As was revealed by Theorem 2.1 and a following discussion in Section 2 , the determination of the strong stabilizability of a given n-D system $p = f/g$ can be reduced to the computation of the winding numbers of the images of cycles in $V(f) \cap \bar{U}^n$ under the map g, each cycle being a representative of a homology class of the first order. One may use the algorithms developed in Schwartz and Sharir (1983) to construct such a set of cycles which represents the first homology group.

Alternatively such a set can also be constructed from the graph composed of the 0-cells and 1-cells of the cad. This can be implemented by a simple procedure: first, construct a *maximal forest* (see, e.g., Foulds, 1992) of the graph; next find the *fundamental cycles* associated with the forest, which contain at least one representative for each homology class of the the first order. As the boundaries of 2-cells are homologous to the 0 cycle, by using the data of the incidence between 2-cells and 1-cells, the set of the fundamental cycles can be reduced to contain exactly one representative for each homology class. This is a typical *algebraic simplification* problem (Buchberger and Loos, 1982).

Assuming the first homology cycles have been constructed from 0-cells and 1-cells, in the following we develop algorithms for computing the corresponding winding numbers.

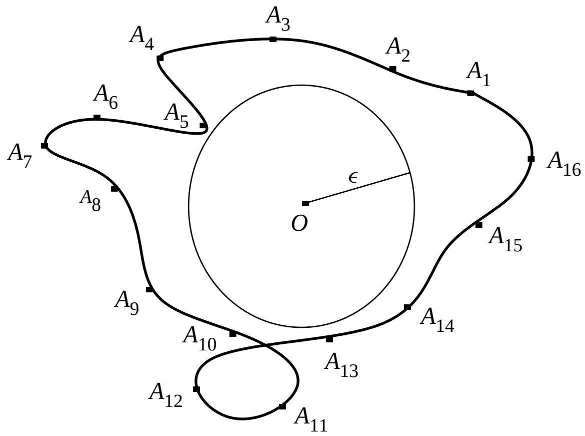

Figure 2. Discrete representation of a cycle $g(\gamma)$ in $\mathbb{C}^*$.

3.2. Computation of the Winding Number

Given a cycle on $V(f) \cap \bar{U}^n$, for computing the winding number of its image under the map g in $\mathbb{C}^*$, we develop a discrete symbolic algorithm.

Let $g(\gamma)$ be the image of a closed cycle mapped by g into $\mathbb{C}^*$. Let ϵ be a real number such that $\epsilon > 0$ and

$$|g(\gamma(t))| > \epsilon, \qquad \forall t \in [0, 1]. \tag{3.2}$$

This means that the distance from the origin to the cycle $g(\gamma)$ is greater than ϵ.

Suppose we have found a finite set of points $A_1, \dots, A_N$ on $g(\gamma)$ which divide $g(\gamma)$ into N segments such that for each segment A_jA_{j+1}, $j = 1, \dots, N$ ($A_{N+1} = A_1$), for any point P on A_jA_{j+1}, we have $|P - A_j| < \epsilon$ and $|P - A_{j+1}| < \epsilon$. We call such a point set $\{A_1, \dots, A_N\}$ a **discrete representation** of the cycle. See Figure 2.

Let $\angle A_jOA_{j+1}$ denote the angle measured counterclockwise from the vector $\vec{OA_j}$ to $\vec{OA_{j+1}}$, where O is the origin of the $\mathbb{C}$-plane. Now we have

$$\int_{A_jA_{j+1}} \frac{d\tau}{\tau} = i\angle A_jOA_{j+1} + \log|A_{j+1}| - \log|A_j|. \tag{3.3}$$

The winding number of $g(\gamma)$ is

$$W(g(\gamma)) = \frac{1}{2\pi i}\int_{g(\gamma)} \frac{d\tau}{\tau} = \frac{1}{2\pi i}\sum_{j=1}^{N}\int_{A_jA_{j+1}} \frac{d\tau}{\tau} = \frac{1}{2\pi}\sum_{j=1}^{N} \angle A_jOA_{j+1}. \tag{3.4}$$

As $|A_{j+1} - A_j| < \epsilon$, we have

$$-\frac{\pi}{3} < \angle A_jOA_{j+1} < \frac{\pi}{3}. \tag{3.5}$$

We use the notation $A_j \prec A_{j+1}$ to denote the condition that $0 \le \angle A_jOA_{j+1} < \pi$.

Let $\Im(\frac{A_{j+1}}{A_j})$ designate the imaginary part of the complex number $\frac{A_{j+1}}{A_j}$. Clearly,

$$A_j \prec A_{j+1} \qquad \text{iff } \Im(\frac{A_{j+1}}{A_j}) \ge 0. \tag{3.6}$$

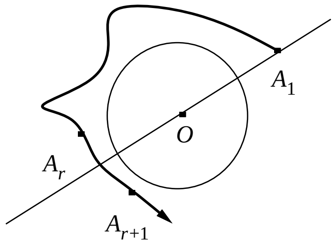

Figure 3. $W_r = 0$, sign(θ_r)$= +$; $W_{r+1} = 1$, sign(θ_{r+1})$= -$.

For $1 \le r \le N$, let

$$\sum_{j=1}^{r} \angle A_j O A_{j+1} = 2W_r \pi + \theta_r, \tag{3.7}$$

where W_r is an integer and $-\pi < \theta_r \le \pi$. Define

$$sign(\theta_r) = \begin{cases} + & \text{if } \theta_r \ge 0, \\ - & \text{if } \theta_r < 0. \end{cases} \tag{3.8}$$

Clearly,

$$sign(\theta_r) = \begin{cases} + & \text{if } \Im(\frac{A_r}{A_1}) \ge 0, \\ - & \text{otherwise.} \end{cases} \tag{3.9}$$

See Figure 3.

The winding number can be computed by the following symbolic procedure without calculating the exact numerical values of each term in expression (3.4).

Procedure for computing the winding number:

Step 1: Set $W_0 = 0$ and $sign(\theta_0) = +$;
Step 2: For $r = 1$ to N,
Compute $sign(\theta_{r+1})$;
Compute W_{r+1} from W_r, $sign(\theta_r)$ and $sign(\theta_{r+1})$;
Step 3: Winding number of $g(\gamma) = W_{N+1}$.

In Step 2, W_{r+1} can be computed from the following table.

	(θ_r, θ_{r+1})	Change of winding number
Case 1	$(+,+)$	$W_{r+1} = W_r$
Case 2	$(+,-)$	if $A_r \prec A_{r+1}$ then $W_{r+1} = W_r + 1$, else $W_{r+1} = W_r$
Case 3	$(-,-)$	$W_{r+1} = W_r$
Case 4	$(-,+)$	if $A_{r+1} \prec A_r$ then $W_{r+1} = W_r - 1$, else $W_{r+1} = W_r$

Case 1 and case 3 are obvious. In case 2, the condition $A_r \prec A_{r+1}$ implies that $\vec{OA_r}$ rotates counterclockwise to $\vec{OA_{r+1}}$ by an angle $< \pi$, and we must have $W_{r+1} = W_r + 1$ to have an increase of the total angle, as shown in Figure 4 (a). Otherwise $\vec{OA_r}$ rotates clockwise to $\vec{OA_{r+1}}$ to yield a decrease in the total angle, and we have $W_{r+1} = W_r$, as shown in Figure 4 (b). Similar explanation applies to case 4.

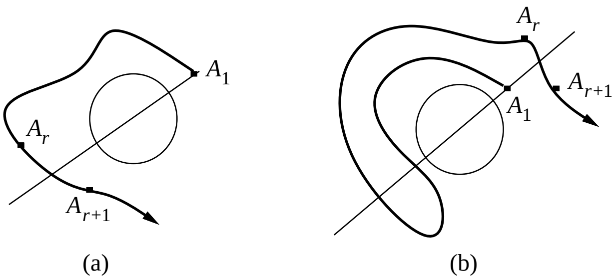

Figure 4. Case 2 in Step 3 for computing winding number: (a) $A_r \prec A_{r+1}$ ($W_{r+1} = W_r + 1$); (b) $A_{r+1} \prec A_r$ ($W_{r+1} = W_r$).

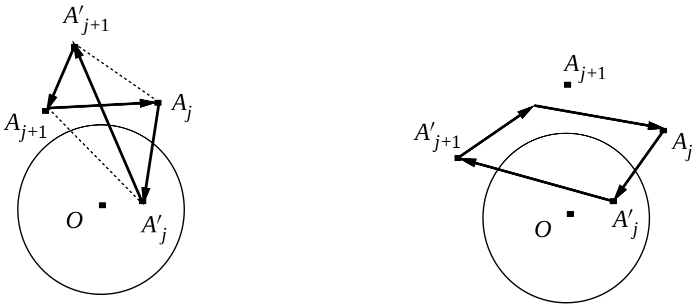

Figure 5. The polygon spanned by A'_j, A'_{j+1}, A_{j+1} and A_j does not contain the origin.

The computation of $sign(\theta_{r+1})$ and the determination of whether $A_r \prec A_{r+1}$ holds or not involved in the above procedure can be reduced to computing the signs of $\Im(\frac{A_{r+1}}{A_1})$ and $\Im(\frac{A_{r+1}}{A_r})$, respectively, and therefore can be executed by arithmetic operations on the complex numbers A_r. If the A_r values are rational complex numbers, the procedure can be executed by arithmetic operations on rational numbers.

Proposition 3.1. *Suppose $A'_1, \ldots, A'_N$ are N points in $\mathbb{C}^*$ such that $|A'_j - A_j| < \epsilon$, then the winding number of $g(\gamma)$ can be computed by substituting A'_j for A_j in the above procedure.*

Proof. As

$$|A_j - A_{j+1}| < \epsilon, |A_j - A'_j| < \epsilon \qquad \text{and} \qquad |A_{j+1} - A'_{j+1}| < \epsilon,$$

the polygon spanned by the four points A'_j, A'_{j+1}, A_{j+1} and A_j does not enclose the origin, as shown in Figure 5.

We have

$$\angle A'_j O A'_{j+1} + \angle A'_{j+1} O A_{j+1} + \angle A_{j+1} O A_j + \angle A_j O A'_j = 0, \tag{3.10}$$

$$\angle A'_j O A'_{j+1} - \angle AjOA_{j+1} = \angle A_{j+1} O A'_{j+1} - \angle A_j O A'_j.$$

$$\sum_{j=1}^{N} \angle A_j O A_{j+1} - \sum_{j=1}^{N} \angle A'_j O A'_{j+1} = \sum_{j=1}^{N} \angle A_{j+1} O A'_{j+1} - \sum_{j=1}^{N} \angle A_j O A'_j = 0.$$

Note that $A_{N+1} = A_1$. The winding number, therefore, is

$$\frac{1}{2\pi}\sum_{j=1}^{N} \angle A_j O A_{j+1} = \frac{1}{2\pi}\sum_{j=1}^{N} \angle A'_j O A'_{j+1}. \ \square \tag{3.11}$$

This proposition gives the condition on the error limit allowable in tracing the cycles in order to compute their winding numbers. If the A'_j's are chosen to be rational complex numbers, the winding numbers can be computed by simple arithmetic operations.

Given a cycle γ on $V(f) \cap \bar{U}^n$, the following procedure gives a method for dividing $g(\gamma)$ into a finite number of segments so that the previous algorithm can be applied to compute $W(g(\gamma))$.

Procedure for finding a discrete representation for a given cycle

For an n-tuple $z = (z_1, \cdots, z_n)$ of complex numbers, we define $||z||_\infty = max_j\{|z_j|,\ 1 \le j \le n\}$.

Subroutine I: Find $\epsilon > 0$, s.t.

$$|g(z)| > \epsilon \ \forall z \in V(f) \cap \bar{U}^n; \tag{3.12}$$

Subroutine II: Find $\delta > 0$, s.t. $\forall z,\ z + \Delta z \in V(f) \cap \bar{U}^n$, if $||\Delta z||_\infty < \delta$, then

$$|g(z + \Delta z) - g(z)| < \epsilon; \tag{3.13}$$

Subroutine III: For each 1-cell of γ, divide it into a finite number of segments $Q_1Q_2, Q_2Q_3, \ldots, Q_{m-1}Q_m$, s.t. for $j = 1, \ldots, m-1$,

$$\forall z \in Q_jQ_{j+1}, ||z - Q_j||_\infty < \delta,\ ||z - Q_{j+1}||_\infty < \delta. \tag{3.14}$$

Subroutine I has to solve an extreme value problem under polynomial constraints. Besides conventional optimization methods, there is also an approach based on the cylindrical algebraic decomposition for solving the problem (Weispfenning, 1997). Let $g' = |g(z)| - x$ be a real polynomial in the real coordinates of z and a new variable x. Let

$$\Phi' = \{f(z), |z_1| - 1, \ldots, |z_n| - 1,\ |g(z)| - x\}, \tag{3.15}$$

a set of real polynomials in real coordinates. We construct an Φ'-invariant cad of the space R^{2n+1} with x as the last base dimension. Consider the cells satisfying $f(z) = 0, |z_1| - 1 \le 0, \ldots, |z_n| - 1 \le 0$ and $|g(z)| - x < 0$; their projection to the x-coordinate axis are a collection of intervals. Let ϵ be the minimum of the left terminals of these intervals. It is easy to see that if z satisfies $f(z) = 0, |z_1| - 1 \le 0, \ldots, |z_n| - 1 \le 0$, then $|g(z)| > \epsilon$.

In Subroutine II, for $j = 1, \ldots, n$, let M_j be the sum of the absolute values of the coefficients of the polynomial $\frac{\partial g}{\partial z_j}(z)$; let $M = max\{M_1, \ldots, M_n\}$. For any $z, z+\Delta z \in \bar{U}^n$, we have

$$\begin{aligned} |g(z + \Delta z) - g(z)| &= \left| \int_z^{z+\Delta z} dg \right| = \left| \sum_{j=1}^{n} \int_z^{z+\Delta z} \frac{\partial g}{\partial z_j} dz_j \right| \\ &\le \sum_{j=1}^{n} \left| \int_z^{z+\Delta z} \frac{\partial g}{\partial z_j} dz_j \right| \le \sum_{j=1}^{n} M_j |\Delta z_j| \le nM||\Delta z||_\infty. \end{aligned}$$

δ can then be chosen to be $\frac{\epsilon}{nM}$.

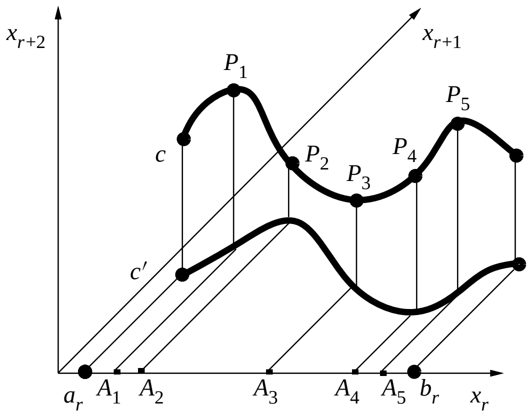

Figure 6. Critical points on a cell c. The cell c' is the projection of c, the projection of c' is a segment $[a_r, b_r]$ on the x_r-axis.

Subroutine III identifies $V(f) \cap \bar{U}^n$ as a semi-algebraic subset of R^{2n} with a proper real coordinate system $\{x_1, \ldots, x_{2n}\}$, which implies that a cad has already been constructed. We first divide the 1-cell by some *critical points*, to be defined below, so that the resulting segments have a certain monotone property (see Figure 6).

Consider a 1-cell defined by:

$$\begin{cases} x_1 = a_1, \\ \ldots, \\ x_{r-1} = a_{r-1}; \\ a_r \le x_r \le b_r; \\ F_{r+1}(a_1, \ldots, a_{r-1}, x_r, x_{r+1}) = 0, \\ \ldots, \\ F_{2n}(a_1, \ldots, a_{r-1}, x_r, x_{r+1}, \ldots, x_{2n}) = 0. \end{cases}$$

We have

$$\begin{cases} \frac{dx_{r+1}}{dx_r} &= -(\frac{\partial F_{r+1}}{\partial x_r})/(\frac{\partial F_{r+1}}{\partial x_r}) = \frac{G_{r+1}}{D_{r+1}}, \\ & \ldots, \\ \frac{dx_{2n}}{dx_r} &= -(\frac{\partial F_{2n}}{\partial x_r} + \frac{\partial F_{2n}}{\partial x_{r+1}}\frac{dx_{r+1}}{dx_r} + \cdots + \frac{\partial F_{2n}}{\partial x_{2n-1}}\frac{dx_{2n-1}}{dx_r})/(\frac{\partial F_{2n}}{\partial x_{2n}}) \\ &= \frac{G_{2n}}{D_{2n}}. \end{cases} \tag{3.16}$$

Let $A_1 < A_2 < \cdots < A_k$ be a set of numbers defined as

$$\begin{aligned} \{A_1, \ldots, A_k\} = \{x_r \in [a_r, b_r]| \\ &\exists\ x_{r+1}(G_{r+1} = 0 \wedge F_{r+1} = 0) \vee \\ &\exists\ x_{r+1} \exists x_{r+2}(G_{r+2} = 0 \wedge F_{r+1} = 0 \wedge F_{r+2} = 0) \vee \\ &\quad \ldots \vee \\ &\exists\ x_{r+1} \ldots \exists x_{2n}(G_{2n} = 0 \wedge F_{r+1} = 0 \wedge \ldots \wedge F_{2n} = 0)\}. \end{aligned} \tag{3.17}$$

Let $P_1, \ldots, P_k$ be the points on the cell with x_r-coordinate $A_1, \ldots, A_k$, respectively. These points are the critical points and can be obtained by an algorithm for assigning a *sample point* on a cell in the construction of the cad (Arnon *et al.*, 1984).

It is obvious that on each segment P_jP_{j+1}, the coordinates $x_{r+1}, \ldots, x_{2n}$ are monotone functions in x_r. We call such a segment a *monotone segment*. If $||P_j - P_{j+1}||_\infty \leq \delta$, then P_jP_{j+1} is already a segment satisfying the requirement (3.14) in Subroutine III; otherwise we use the following recursive procedure to subdivide it into finer segments.

Let

$$L = \max\{|(P_j - P_{j+1})_1|, \ldots, |(P_j - P_{j+1})_{2n}|\},$$

where $(P_j - P_{j+1})_k$ is the k-th component of $P_j - P_{j+1}$, $k = 1, \ldots, 2n$. If $L = |(P_j - P_{j+1})_k|$ for some k, then find a point Q on P_jP_{j+1} with $\frac{1}{2}(P_j + P_{j+1})_k$ as its k-th coordinate. If $||P_j - Q||_\infty \leq \delta$, the segment P_jQ meets the requirement of Subroutine III, else repeat an analogous division process until the resulting segments all satisfy the requirement of Subroutine III. The same procedure applies to segment QP_{j+1}.

For a monotone segment, let us call the length of its projection on the x_r-coordinate the r-th length. A segment which has a unit r-th length can be partitioned into segments each with the r-th length $\leq \delta$ by less than $2\frac{1}{\delta}$ bisection operations, $r = 1, \ldots, 2n$. A total number of $4n\frac{1}{\delta}$ operations are needed to divide such a segment into segments with each of the r-th coordinates, $r = 1, \ldots, 2n$, no larger than δ.

When the dimension n is fixed, the number of 1-cells is of polynomial order in the total degree d of f (Collins, 1975, Theorem 12). It can also be easily seen that the number of critical points on any 1-cell is of polynomial order in d.

Both the space complexity and the time complexity of the procedure are therefore $O((\frac{4n}{\delta})P(d)) = O((\frac{4n^2M}{\epsilon})P(d))$, where $P(d)$ is a polynomial in d, the total degree of f.

Remarks.

(1) Complexity issues for constructing the cad are addressed in Collins (1975), and for computing the homology groups are in Schwartz and Sharir (1983).
(2) It is interesting to note that ϵ contributes to the complexity for the determination of strong stabilizability. In the limit case that $\epsilon = 0$, the system is not stabilizable (even by a non-stable compensator).
(3) For a specific curve γ, there may exists an $\epsilon_\gamma > \epsilon$, such that $|g(\gamma(t))| > \epsilon_\gamma$, $\forall t \in [0, 1]$. In this case, the requirement on a discrete representation $\{A_1, \ldots, A_N\}$ for $g(\gamma)$ may be weakened as $\forall P \in A_iA_{i+1}, |P - A_j| < \epsilon_\gamma$ and $|P - A_{j+1}| < \epsilon_\gamma$. Obviously, the larger ϵ_γ is than ϵ, the easier such a discrete representation is obtained. However, estimating ϵ_γ for each γ may be more expensive than just estimating a single ϵ.

3.3. sign test

In order to decide whether a real system $p = f/g$ has a stable real compensator or not, we have to compute the sign of g on each self-conjugate connected component of $V(f) \cap \bar{U}^n$.

general case

Given a connected component X, if it has a real vertex, then it is self-conjugate and the sign of g can be directly evaluated; otherwise let z_0 be a complex vertex (0-cell) on X. In general, the conjugate $\overline{z_0}$ is not necessarily a vertex of the cad. Suppose $\overline{z_0}$ is in some cell c. Find a vertex v which is a face of c, i.e. v lies in the topological closure of c

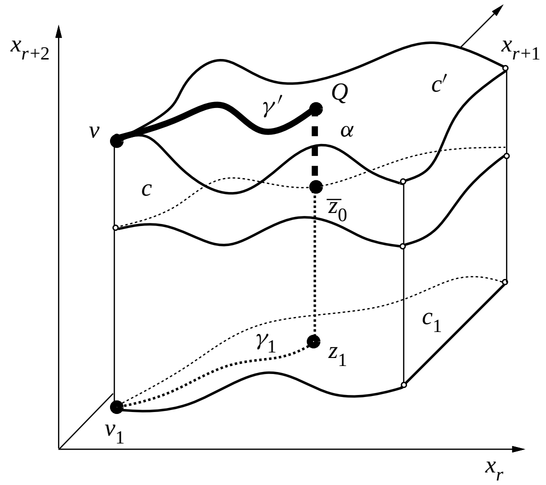

Figure 7. c is a sector over c_1. c' is a section on the top of c. γ_1 is a path in c_1. γ' is the *lifting-up* of γ_1 to c'. $\gamma = \gamma'\alpha$ is a path in c connecting v to $\bar{z_0}$.

(Cooke and Finney, 1967; Schwartz and Sharir, 1983). If there exists no chain composed of 1-cells which connects z_0 to v, then X and $\overline{X}$ are not self-conjugate and we no longer need to consider them. Otherwise X is self-conjugate and we can find a path β which connects z_0 to v. This can be realized with any standard graph theoretic algorithm. See, e.g., Foulds (1992).

It remains to find a path γ in c which connects v to $\overline{z_0}$; and then decide the parity of the integer m which satisfies

$$\int_{\gamma_0} \frac{dg}{g} = \overline{G_0} - G_0 + m2\pi i,$$

where G_0 is a number s.t. $e^{G_0} = g(z_0)$, and $\gamma_0 = \gamma\beta$.

Let v_1 and z_1 be the projection of v and $\overline{z_0}$, respectively, to the R^{2n-1} space. Let c_1 be the base cell of c in R^{2n-1}. Suppose we have already found a path γ_1 in c_1 which connects v_1 to z_1. See Figure 7.

If c is a *section* (Arnon *et al.*, 1984) defined by a polynomial F over c_1, then γ is simply the *lifting-up* of γ_1 defined as

$$\begin{aligned}\gamma = \{&(x_1, \ldots, x_{2n}) \in R^{2n} | \\ &(x_1, \ldots, x_{2n-1}) \in \gamma_1,\ F(x_1, \ldots, x_{2n}) = 0\}.\end{aligned} \tag{3.18}$$

Otherwise if c is a *sector* over c_1, as illustrated in Figure 7, then let c' be a *section* (Arnon *et al.*, 1984) incident to c over c_1, defined by polynomial F', and let Q be the point in c' which is projected to z_1 in R^{2n-1}. Let γ' be the lifting-up of γ_1 up to c', as is defined by

$$\begin{aligned}\gamma' = \{&(x_1, \ldots, x_{2n}) \in R^{2n} | \\ &(x_1, \ldots, x_{2n-1}) \in \gamma_1,\ F'(x_1, \ldots, x_{2n}) = 0\}.\end{aligned} \tag{3.19}$$

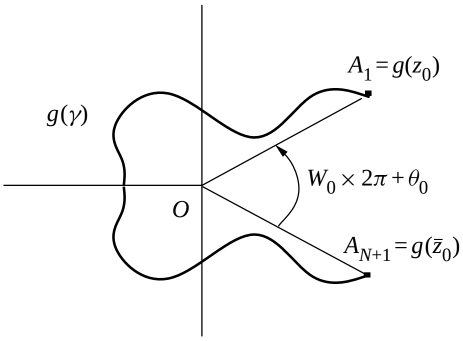

Figure 8. $\int_{\beta+\gamma} \frac{dg}{g} + G_0 - \bar{G}_0$ can be computed by using the winding number algorithm.

Let α be the straight line segment connecting Q and $\overline{z_0}$, then $\gamma = \alpha\gamma'$ is a path in c connecting v to $\overline{z_0}$.

γ_1 can be recursively constructed by projection into lower dimensional spaces.

By the procedure in Subroutine III described above, a discrete representation of $g(\gamma)$ can be found.

Remark. Finding a path connecting two given points is similar to the *path planning* problem. The reader is referred to Schwartz and Sharir (1983, Section 3) for a related discussion.

For a similar discussion as at the end of Section 3.2, it is easy to see that the complexity of the procedure is again $O((\frac{4n^2M}{\epsilon})P(d))$, $P(d)$ is a polynomial in d, the total degree of f.

We are now ready to compute

$$\int_{\gamma\beta} \frac{dg}{g} + G_0 - \overline{G_0}. \tag{3.20}$$

Now $\frac{1}{i}(G_0 - \overline{G_0})$ is actually the angle formed by rotating $Og(\overline{z_0})$ to $Og(z_0)$. Let $A_1 = g(z_0), A_2, \ldots, A_N, A_{N+1} = g(\overline{z_0})$ be a discrete representation for the chain $\beta + \gamma$; Write

$$\angle g(\overline{z_0})Og(z_0) = W_0 2\pi + \theta_0, \tag{3.21}$$

where W_0 is an integer and $-\pi < \theta_0 \leq \pi$. The integers m can be obtained by the procedure of Section 3.2 for computing the winding number, simply by substituting the above initial values in the step 1 of the procedure. See Figure 8.

SPECIAL CASE

For a vertex z_0, if $\overline{z_0}$ is also a vertex, then either there exists a path composed of 1-cells in the cad that connects z_0 to $\overline{z_0}$, or the two components containing z_0 and $\overline{z_0}$, respectively, are disjoint and are not self-conjugate. In either case, we will not need to execute the above procedure for finding a path connecting an inner point within some cell to a vertex which is a face of the cell.

4. Example

We illustrate the algorithm for computing the winding number with an example. Let $f = 1 - 4z_1z_2$, $g = z_1$, as in Example 1 in Section 2.

Rewrite f in real coordinates:

$$f = 1 - 4(x_1 + iy_1)(x_2 + iy_2)$$

$$= 1 - 4x_1x_2 + 4y_1y_2 - i4(x_1y_2 + x_2y_1).$$

$V(f) \cap \bar{U}^2$ can be considered as a semi-algebraic set in $\mathbb{R}^4$ defined by the following formulae:

$$1 - 4x_1x_2 + 4y_1y_2 = 0, \tag{i}$$
$$x_1y_2 + x_2y_1 = 0, \tag{ii}$$
$$x_1^2 + y_1^2 - 1 \leq 0, \tag{iii}$$
$$x_2^2 + y_2^2 - 1 \leq 0. \tag{iv}$$

For simplicity of presentation, instead of following exactly a standard cylindrical algebraic decomposition procedure, we use an informal method to decompose $V(f) \cap \bar{U}^2$ into a cell complex structure in the essence of a cad.

Eliminating y_2 in (i) and (ii), we have

$$x_1(4x_1x_2 - 1) + 4x_2y_1^2 = 0,$$

$$x_2 = \frac{x_1}{4(x_1^2 + y_1^2)},$$

for $x_1^2 + y_1^2 \neq 0$, as is ensured by equation (i). This says that the projection in $\mathbb{R}^3$ is a surface parametrized by x_1 and y_1. From equations (i) and (ii) it can be deduced that

$$16(x_1^2 + y_1^2)(x_2^2 + y_2^2) = 1,$$

therefore the inequalities (iii) and (iv) can be rewritten as

$$1/16 \leq x_1^2 + y_1^2 \leq 1.$$

The projection of the 1-cells into the (x_1, y_1)-plane form two cycles ,

$$x_1^2 + y_1^2 = 1/16 \qquad \text{and} \qquad x_1^2 + y_1^2 = 1.$$

The semi-algebraic set $V(f) \cap \bar{U}^2$ is a strip bounded by the two cycles. See Figure 9 (a). A cell complex of it consists of eight 0-cells (points), 12 1-cells and four 2-cells. See Figure 9 (b).

Let γ be the curve described by the equation $x_1^2 + y_1^2 = 1$. The 1-cells of γ have the following parametric representation with respect to x_1, which are derived from equations (i) and (ii).

$$y_1 = \pm\sqrt{1 - x_1^2}, \tag{4.1}$$
$$x_2 = \frac{1}{4}x_1, \tag{4.2}$$
$$y_2 = \frac{4x_1x_2 - 1}{4y_1}. \tag{4.3}$$

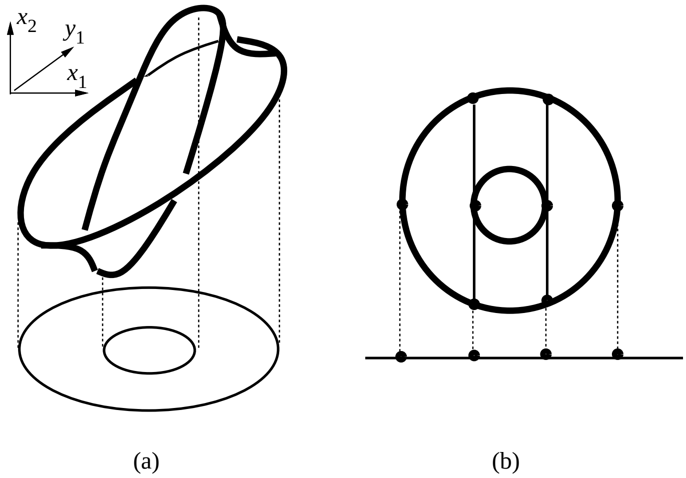

Figure 9. (a) Projections of $V(f) \cap \bar{U}^n$ into the (x_1, y_1, x_2)-space and (x_1, y_1)-plane. (b) Projection of a cad in the (x_1, y_1)-plane.

A discrete representation for γ can be constructed by Subroutines I, II and III of Section 3.2 as follows.

For Subroutine I, suppose by some optimization method we have found $\epsilon_\gamma = 0.8$, such that $\forall |g(\gamma)| > \epsilon_\gamma$ (cf. Remark 3 at the end of Section 3.2).

For Subroutine II, we have $M = 1$, $\sigma = \frac{\epsilon}{2M} = 0.4$.

For Subroutine III, the derivatives of the coordinates y_1, x_2, y_2 with respect to x_1 can be computed as follows from (4.1) and (4.2).

$$\frac{dy_1}{dx_1} = \mp \frac{x_1}{\sqrt{1 - x_1^2}}, \tag{4.4}$$

$$\frac{dx_2}{dx_1} = \frac{1}{4}, \tag{4.5}$$

$$\frac{dy_2}{dx_1} = \frac{x_1(1 - x_1^2)}{4y_1^3}. \tag{4.6}$$

Clearly, $x_1 = 0$ is the only possible x_1-coordinate for a critical point on a cell of γ. Therefore, the curve segments partioned by a total of 8 points, shown in Figure 10(a), are all monotone in the sense defined in Section 3.2 and can be refined by a bisection procedure to obtain a discrete representation. For instance, a bisection operation applied to segment P_1P_2 with respect to the x_1-coordinate yields a point Q_1, a further bisection with respect to the y_1-coordinate of P_1Q_1 yields Q_2, as shown in Figure 10(b). This partition is fine enough to satisfy the requirement for a discrete representation. As a result, a discrete representation with 16 points is obtained as shown in Figure 11. Note that because $\sigma = 0.4$, an error of up to 0.4 is allowable for each coordinate of each point. Note also that in the example $g(z) = z_1$, the image of γ under the map g is identical to the projection in the (x_1, y_1)-plane.

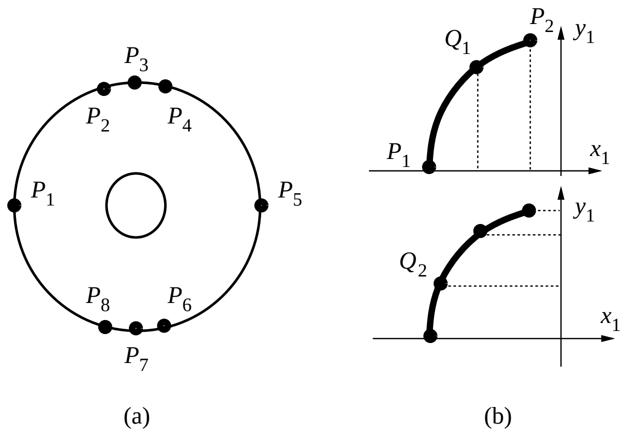

Figure 10. (a) 1-cells and critical points. (b) Bisection procedure applied to cell P_1P_2.

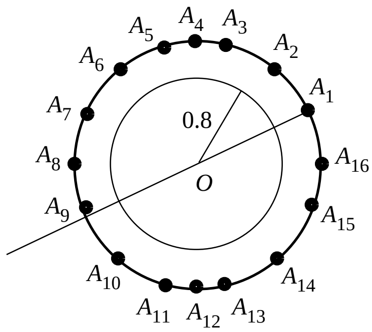

Figure 11. A discrete representation for $g(\gamma)$ with 16 points.

A record of the process for computing the winding number by the procedure of Section 3.2 is as follows.

$$\theta_0 = \theta_1 = \cdots = \theta_9 = +, \theta_{10} = \cdots = \theta_{16} = -, \theta_{17} = +;$$
$$W_0 = W_1 = \cdots = W_9 = 0, W_{10} = \cdots = W_{16} = 1, W_{17} = 1.$$

The winding number is equal to 1.

5. Conclusion

Computational procedures based on the cylindrical algebraic decomposition of semi-algebraic sets have been presented for determining the strong stabilizability of linear n-D systems. The essential part is the computation of winding numbers of cycles on semi-algebraic sets in a Euclidean space. For this we presented a symbolic algorithm that computes the winding number from an approximate discrete representation of the cycle. Computational procedures for constructing such a discrete representation have also been presented.

Unfortunately, our procedures are still too expensive to be implemented at the present stage. Nevertheless, our results do provide a computational method for solving the strong stabilizability test problem for linear n-D systems. By properly applying recent computer algebra packages, it is possible to obtain a solution for this problem. As a large variety of practical control systems: time-delay systems, distributed system, etc. can be described as n-D systems (Bose, 1985), we believe the proposed method is of practical importance.

Recent application of computer algebra to decision problems in control system have been limited to those problems which have a quantifier elimination formulation. Our research has shown the potential application of computer algebra methods to a wider range of control system decision problems.

We conclude this paper with some remarks on two open problems:

(i) In this paper we have not addressed the strong stabilizability of *multi-input multi-output* (MIMO) n-D systems (Xu *et al.*, 1994; Lin, 1998) which are described by transfer matrices in several variables. We believe that this may as well be reduced to a topological problem and may be solved by analogous methods.

(ii) In Appendix A we used the Gröbner bases method for finding a stabilizing compensator, which happens to be stable itself. Unfortunately, this is not always the case, as was demonstrated in Example 3 in Section 2.2. The development of a general algorithm for finding a stable compensator remains to be an interesting open problem.

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Appendix A. Finding a Stabilizer

The method is simple enough to be stated here.

$$V(f) \cap V(g) = \left\{ \left(-\frac{1+\sqrt{2}}{2}, \frac{1-\sqrt{2}}{2} \right), \left(\frac{-1+\sqrt{2}}{2}, \frac{1+\sqrt{2}}{2} \right) \right\}.$$

Let

$$s(z_1, z_2) = \left(z_1 + \frac{1+\sqrt{2}}{2} \right) \left(z_2 - \frac{1+\sqrt{2}}{2} \right).$$

We have

$$V(s) \supset V(f) \cap V(g).$$

The ideal generated by the three polynomials f, g and $1 - z_3 s$ in $\mathbb{C}[z_1, z_2, z_3]$ contains the unit 1. Using Gröbner bases method, we can obtain

$$1 = -\frac{1}{4} z_3 f(z_1, z_2) - \frac{1+\sqrt{2}}{2} z_3 g(z_1, z_2) + (1 - z_3 s(z_1, z_2)).$$

Substituting z_3 with $\frac{1}{s}$, we have

$$f + 2(1+\sqrt{2})g = -4s.$$

As

$$V(s) \cap \bar{U}^2 = \emptyset,$$

$h = \frac{1}{2(1+\sqrt{2})}$ is indeed a stable stabilizer.

Appendix B. A Note on the Concept of Sign

Let $z_0 \in X$ be a real point. For a point $z \in \overline{X}$, $\overline{z} \in X$, there exists a path γ in $\overline{U}^n \cap V(f)$ that connects z_0 to $\overline{z}$. Obviously, $\overline{\gamma}$ connects $\overline{z_0} = z_0$ to z, so $z \in X$.

Suppose $g(z_0) > 0$. Then $\exists G_0 \in \mathbb{R}$, $e^{G_0} = g(z_0)$. Let

$$G(z) = G_0 + \int_\gamma \frac{dg(\tau)}{g(\tau)}, \qquad z \in X,$$

where γ is any curve which connects z_0 to z in X.

We have $e^{G(z)} = g(z)$ and

$$\overline{G(z)} = G_0 + \int_{\overline{\gamma}} \frac{dg(z)}{g(z)} = G(\overline{z}).$$

This shows the conventional positivity of $g(z_0)$ coincides with that of our definition; the negativity part is then obvious, too.

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